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On UrKdV and UrKP

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Abstract

We present two extensions of Wilson's explanation of the Miura map from MKdV to KdV. In the first we explain the map of Svinolupov *et.al.* from a certain UrKdV-like equation to KdV, and in the second we explain Konopelchenko's map from the modified KP equation to KP. In the course of the latter we introduce an "UrKP" system, with an infinite dimensional symmetry, providing us with a systematic method to construct Bäcklund transformations for the modified KP and KP equations.

Résumé

Nous donnons deux généralisations du travail présenté par Wilson sur l'explication de la carte de Miura de l'équation de mKdV à celle de KdV. La première généralisation justifie la carte donnée par Svinolupov *et.al.* d'une certaine équation reliée à l'équation de UrKdV à celle de KdV elle-même. Ensuite, comme seconde généralisation, nous expliquons la carte presentée par Konopelchenko sur le passage de l'équation de KP modifiée à celle de KP. Dans cette seconde partie, nous introduisons un système que nous baptisons "UrKP", qui possède une symétrie de dimension infinie et qui nous fournit une méthode systématique pour dériver les transformations de Bäcklund pour les équations de KP modifiée et de KP.

1.Introduction

Probably the most important insight in the theory of integrable systems in recent years is Wilson's explanation of the Miura map [1]. If j satisfies the MKdV equation

$$j_t = \tfrac{1}{4} j_{xxx} - \tfrac{3}{8} j^2 j_x \ , \tag{1}$$

then $u = -\frac{1}{2}(j_x - \frac{1}{2}j^2)$ solves the KdV equation

$$u_t = \tfrac{1}{4} u_{xxx} - \tfrac{3}{2} u u_x \ . \tag{2}$$

Now in general if v is some function of j and its x-derivatives, we can compute v_t using (1), but we should not expect to be able to write v_t as a function of v and its x-derivatives. As Wilson explains, the fact that u_t *can* be written in terms of u and its x-derivatives suggests strongly that (1) has some symmetry, and that u is invariant under this symmetry. To see that this is indeed the case it is necessary to introduce the UrKdV equation

$$q_t = \tfrac{1}{4}\left(q_{xxx} - \frac{3q_{xx}^2}{2q_x} \right) \ . \tag{3}$$

Via the map $j = q_{xx}/q_x$, a solution of UrKdV generates a solution of MKdV, which in turn generates, via the Miura map, a solution $u = -\frac{1}{2}(q_{xxx}/q_x - 3q_{xx}^2/2q_x^2)$ of KdV. Now (3) can be seen to be invariant under the group of Möbius transformations $q \to (aq+b)/(cq+d)$, $ad - bc = 1$. The reason j_t can be written in terms of j and its derivatives is that j is (the in some sense unique) invariant under the $c = 0$ subgroup of Möbius transformations; similarly the reason u_t can be written in terms of u and its derivatives is that u is (the in some sense unique [2]) invariant under the full group of Möbius transformations. The Miura map, which gives u in terms of j, reflects the fact that an invariant of the full group is of course an invariant of any subgroup; but since the $c = 0$ subgroup of Möbius transformations is not a normal subgroup of the full group, the symmetry of MKdV which "explains" the Miura map is not a group symmetry, and cannot be properly explained without introducing UrKdV.

In [3], one of us noted that $\tilde{j} = q_{xx}/q_x - 2q_x/q$ is (the in some sense unique) invariant under $b = 0$ Möbius transformations, and $\tilde{j}$ also satisfies MKdV. From the relation between j and $\tilde{j}$ we deduce the x-part of a (strong) Bäcklund transformation for MKdV; if j satisfies MKdV, so does $j - 2/r$, where r satisfies

$$\begin{gathered} r_x + rj = 1 \ , \\ r_t + \tfrac{1}{4} r (j_{xx} - \tfrac{1}{2} j^3) = \tfrac{1}{4}(j_x - \tfrac{1}{2} j^2) \ . \end{gathered} \tag{4}$$

In greater generality, any equation which arises from a group invariant equation as the equation satisfied by a quantity invariant under a (proper) subgroup, will display Bäcklund transformations. This is a powerful way to construct Bäcklund transformations, but such transformations in general give rise to only rather limited classes of solutions.

From what we have said above, it is apparent that the the Miura map and the Bäcklund transformation (4) are purely consequences of the Möbius invariance of the UrKdV equation (3). We could have started with any evolution equation of the form

$$q_t = q_x F(u, u_x, u_{xx}, ...) \ , \tag{5}$$

and deduced 1) that the Miura map maps solutions of

$$j_t = \partial_x(\partial_x + j)F \tag{6}$$

to solutions of

$$u_t = (-\tfrac{1}{2}\partial_x^3 + u\partial_x + \partial_x u)F \ , \tag{7}$$

and 2) that if j satisfies (6) so does $j - 2/r$, where

$$\begin{aligned} r_x + rj &= 1 \ , \\ r_t + r(F_x + jF) &= F \ . \end{aligned} \tag{8}$$

In particular the Miura map and the Bäcklund transformation (4) of MKdV are *not* reflections of the integrability of KdV and its relatives, as is often argued.

The original aim of the current work was to extend Wilson's ideas to explain the KP-Miura map [4], namely that a solution of MKP

$$j_t = \tfrac{1}{4}j_{xxx} - \tfrac{3}{8}j^2 j_x + \tfrac{3}{4}(\partial_x^{-1} j_{yy} - j_x \partial_x^{-1} j_y) \tag{9}$$

gives a solution of KP

$$u_t = \tfrac{1}{4}u_{xxx} - \tfrac{3}{2}uu_x + \tfrac{3}{4}\partial_x^{-1} u_{yy} \tag{10}$$

via $u = -\frac{1}{2}(j_x - \frac{1}{2}j^2 - \partial_x^{-1} j_y)$. The resulting theory turns out to be quite rich, and we will give only a part of it here (section 3). But *en route* to this theory we observed that Wilson's ideas actually have an extension for the KdV system that seems of some importance. It has been observed [5] that for arbitrary constants A, B, a solution of the equation

$$\phi_t = \tfrac{1}{4}\left(\phi_{xxx} - \frac{3\phi_{xx}^2}{2\phi_x} + \frac{3(B^2 - 4A\phi)}{2\phi_x}\right) \tag{11}$$

gives a solution of KdV (2) via the map

$$u = \tfrac{1}{2}\left(\frac{\phi_{xxx}}{\phi_x} - \frac{\phi_{xx}^2}{2\phi_x^2} + \frac{B^2 - 4A\phi}{2\phi_x^2}\right) \ . \tag{12}$$

In section 2 we give the group-theoretical explanation of this map. Setting $A = B = 0$ we deduce from (12) that there is a second map from UrKdV to KdV in addition to the standard "Schwartzian derivative" one given after (3). The existence of these two maps from UrKdV to KdV is equivalent to the $j \to -j$ symmetry of MKdV. It is precisely this feature of MKdV and the higher equations in the MKdV hierarchy that make them unique amongst the equations of form (6) [6].

2.Extension of the UrKdV formalism

As is well known, the KdV equation has a zero curvature formulation:

$$\partial_t M - \partial_x P + [P, M] = 0 \ , \tag{13}$$

where

$$P = \begin{pmatrix} -\frac{1}{4}u_x & -\frac{1}{2}u \\ \frac{1}{4}u_{xx} - \frac{1}{2}u^2 & \frac{1}{4}u_x \end{pmatrix} \quad , \quad M = \begin{pmatrix} 0 & 1 \\ u & 0 \end{pmatrix} \ . \tag{14}$$

P and M are $sl(2)$ matrices. The origin of the UrKdV equation can be understood as follows: (13) is the consistency condition for the equations

$$\begin{aligned} g_t &= -Pg \\ g_x &= -Mg \end{aligned} \tag{15}$$

where g is an $SL(2)$ matrix. The second equation in (15) essentially determines all entries in g in terms of one unknown function q, and gives u in terms of this unknown function q; the first equation in (15) gives the evolution of q. But (15) is invariant under $g \to gh$, where h is a constant $SL(2)$ matrix; it follows that whatever the evolution equation for q is, it will have an SL(2) invariance, and this gives rise to the Möbius invariance of UrKdV.

Similarly, any equation of form (7) can be written in the form (13), with

$$P = \begin{pmatrix} \frac{1}{2}F_x & F \\ Fu - \frac{1}{2}F_{xx} & -\frac{1}{2}F_x \end{pmatrix} \quad , \quad M = \begin{pmatrix} 0 & 1 \\ u & 0 \end{pmatrix} \ , \tag{16}$$

and thus the origin of the Möbius invariant equation (5) can be understood. From the point of view of zero curvature formulations, the feature that distinguishes the equations of the KdV hierarchy from other equations of form (7), is that they have a one-parameter family of zero curvature formulations; for KdV we can take in (13)

$$P = \begin{pmatrix} -\frac{1}{4}u_x & \lambda - \frac{1}{2}u \\ \frac{1}{4}u_{xx} + (\lambda + u)(\lambda - \frac{1}{2}u) & \frac{1}{4}u_x \end{pmatrix} \quad , \quad M = \begin{pmatrix} 0 & 1 \\ u + \lambda & 0 \end{pmatrix} \ , \tag{17}$$

for any λ (λ is usually called the spectral parameter). (13), with the choice (17), can be regarded as the consistency condition for equations of the form (15), but we now take g as a 2×2 matrix which is a formal power series in λ, that is

$$g = \sum_{n=0}^{\infty} g_n \lambda^n , \tag{18}$$

where the g_n are 2×2 matrices[1].

What is now the content of equations (15)? The first row of the second equation constrains g_n to be of the form

$$g_n = \begin{pmatrix} \alpha_n & \beta_n \\ -\alpha_{nx} & -\beta_{nx} \end{pmatrix} , \tag{19}$$

while the second row gives the following relations

$$\begin{aligned} \alpha_{0xx} - u\alpha_0 = \beta_{0xx} - u\beta_0 = 0 \ , \\ \alpha_{n-1} = \alpha_{nxx} - u\alpha_n \ , \quad n > 0 \\ \beta_{n-1} = \beta_{nxx} - u\beta_n \ , \quad n > 0. \end{aligned} \tag{20}$$

For each N we can use the equations (20) for $n \leq N$ to write $\alpha_0, \beta_0, \alpha_1, \beta_1, ..., \alpha_{N-1}, \beta_{N-1}$ in terms of u, α_N, β_N, and we are left with two relations between the three functions u, α_N, β_N. Assuming we can solve these relations and write u in terms of the pair α_N, β_N, which satisfy a single constraint, we can then consider the first equation of (15), which when truncated at order λ^N gives a consistent evolution for the constrained pair α_N, β_N; this evolution must induce the KdV evolution (2) for u. We will compute this evolution for $N = 1$ shortly[2]. But first we note that equations (15) have an infinite dimensional invariance, the invariance $g \to gh$, where h is a matrix valued formal power series in λ. Writing $h = \sum_{n=0}^{\infty} h_n \lambda^n$, we see that if we allow transformations with $\det(h_0) = 0$, we have only a monoid invariance. For ease, we restrict to those h for which $\det(h_0) \neq 0$, to obtain an infinite dimensional group invariance. The dimension of the subgroup that acts nontrivially on any particular g_N, however, is finite. This gives the symmetry group of the evolution equation for the constrained pair α_N, β_N.

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[1] We could restrict g by requiring that $\det(g)$, which can be computed as a formal power series in λ, be 1. We would then only discover the map (12) for $A = 1$, $B = 0$. By rescaling and translating ϕ one can deduce the map for arbitrary $A \neq 0$ and B from this case, but the case $A = 0$ is also of interest.

[2] For higher N it seems one can only solve for u in terms of α_N, β_N in a formal sense, and the evolutions are nonlocal.

To implement the above procedure for $N=1$, it is useful to observe that equations (20) imply that

$$\begin{aligned} \partial_x(\alpha_0\beta_{0x}-\alpha_{0x}\beta_0)=0\ , \\ \partial_x(\alpha_0\beta_{1x}-\alpha_{0x}\beta_1+\alpha_1\beta_{0x}-\alpha_{1x}\beta_0)=0\ . \end{aligned} \tag{21}$$

We therefore define

$$\begin{aligned} A&=\alpha_0\beta_{0x}-\alpha_{0x}\beta_0\ , \\ B&=\alpha_0\beta_{1x}-\alpha_{0x}\beta_1+\alpha_1\beta_{0x}-\alpha_{1x}\beta_0\ . \end{aligned} \tag{22}$$

It can easily be checked that the evolutions for $\alpha_0,\beta_0,\alpha_1,\beta_1$ obtained from (15) imply that $A_t=B_t=0$, and thus A and B are constant. Having introduced A and B, it is now straightforward to deduce from (20) and (22) that

$$u=\frac{(\alpha_1\beta_1'''-\beta_1\alpha_1''')-(\alpha_1'\beta_1''-\beta_1'\alpha_1'')-B}{2(\alpha_1\beta_1'-\beta_1\alpha_1')} \tag{23}$$

(here primes denote differentiation with respect to x), and that the constraint between α_1 and β_1 is

$$A=u'(\alpha_1\beta_1''-\beta_1\alpha_1'')+u^2(\alpha_1\beta_1'-\beta_1\alpha_1')-u(\alpha_1\beta_1'''-\beta_1\alpha_1'''+\alpha_1''\beta_1'-\beta_1''\alpha_1')+(\alpha_1''\beta_1'''-\beta_1''\alpha_1'''). \tag{24}$$

The evolution equations for α_1,β_1 are

$$\begin{aligned} \alpha_{1t}&=\alpha_1'''-\tfrac{3}{4}\alpha_1u'-\tfrac{3}{2}\alpha_1'u\ , \\ \beta_{1t}&=\beta_1'''-\tfrac{3}{4}\beta_1u'-\tfrac{3}{2}\beta_1'u\ . \end{aligned} \tag{25}$$

One can explicitly check that these flows induce the KdV flow for u, and preserve the constraint (24). For completeness we also write down the evolutions of α_0,β_0:

$$\begin{aligned} \alpha_{0t}&=\tfrac{1}{4}\alpha_0u'-\tfrac{1}{2}\alpha_0'u\ , \\ \beta_{0t}&=\tfrac{1}{4}\beta_0u'-\tfrac{1}{2}\beta_0'u\ . \end{aligned} \tag{26}$$

The symmetry group of (25) is given by

$$(\ \alpha_1\quad\beta_1\ )\rightarrow(\ \alpha_0\quad\beta_0\ )h_1+(\ \alpha_1\quad\beta_1\ )h_0\ . \tag{27}$$

The group is a semidirect product of the group of invertible 2×2 matrices under matrix multiplication with the group of 2×2 matrices under matrix addition; elements of the group are pairs (h_0,h_1) of 2×2 matrices, with the product

$$(h_0,h_1).(\tilde{h}_0,\tilde{h}_1)=(h_0\tilde{h}_0,h_0\tilde{h}_1+h_1\tilde{h}_0)\ . \tag{28}$$

Note that A, B transform nontrivially under symmetry group elements: A transforms nontrivially if and only if $det(h_0) \neq 1$; B transforms nontrivially only if $det(h_0) \neq 1$ or $Tr(h_1 h_0^{-1} \neq 0)$.

Following the philosophy of Wilson, we try to construct invariants of subgroups of the symmetry group. α_0, β_0 are invariant under the normal subgroup $\{(h_0, h_1) \mid h_0 = I\}$, and (where it is defined) $q = \beta_0/\alpha_0$ satisfies the UrKdV equation (displaying a group symmetry since we have factored out a normal subgroup). The quantity

$$\phi = \alpha_1 \beta_{1x} - \alpha_{1x} \beta_1 \tag{29}$$

is an invariant under the non-normal subgroup $\{(h_0, h_1) \mid \det(h_0) = 1, h_1 = 0\}$. It is straightforward but tedious to check that ϕ defined thus satisfies (11), and further that u is given in terms of ϕ by (12). The symmetry underlying the existence of this map is thus described; in terms of ϕ it is of course nonlocal.

We conclude this section with a reference to [7]; in this paper a map between a certain coset of a loop group and the space of solutions of a system essentially equivalent to UrKdV is described, and we expect there should be a similar description for certain solutions of the system (25). Also in this paper a somewhat different explanation of the origin of the UrKdV equation is given, based on the zero curvature formulation of MKdV as opposed to KdV.

3.UrKP Formalism

The KP hierarchy has a variety of zero curvature formulations. The one we shall use, which is not the most standard, but might be regarded as the natural extension of equations (13),(17) for KdV, can be inferred from [8]:

$$\partial_t M - \partial_x P + [P, M] = 0 \ , \tag{30}$$

$$P = \begin{pmatrix} -\frac{1}{4}u_x + \frac{3}{4}\partial_x^{-1}u_y & \partial_y - \frac{1}{2}u \\ \frac{1}{4}u_{xx} + \frac{3}{4}u_y + (\partial_y + u)(\partial_y - \frac{1}{2}u) & \frac{1}{4}u_x + \frac{3}{4}\partial_x^{-1}u_y \end{pmatrix} \quad , \quad M = \begin{pmatrix} 0 & 1 \\ u + \partial_y & 0 \end{pmatrix} \ .$$

Here P, M belong to the algebra of finite order 2×2 matrix valued linear differential operators in y. There is in fact a zero curvature formulation of KP using the algebra of finite order $n \times n$ matrix valued linear ordinary differential operators for any n [9]; from each of these one can extract an "UrKP" equation. However here we focus just on the implications of (30).

(30) can be regarded as a consistency condition for the system

$$\begin{aligned} g_t &= -Pg \\ g_x &= -Mg \end{aligned} \tag{31}$$

$$g \in \mathcal{G} = \left\{ g \mid g = \sum_{n=0}^{\infty} g_n(x,y,t)\partial_y^n,\ g_n \in \mathcal{M}_{2,2} \right\}$$

($\mathcal{M}_{2,2}$ denotes the set of 2×2 matrices; $\mathcal{G}$ is a set of formal sums); *alternatively* (30) can be regarded as a consistency condition for the system

$$\begin{aligned} \tilde{g}_t &= \tilde{g}P \\ \tilde{g}_x &= \tilde{g}M \end{aligned} \tag{32}$$

$$\tilde{g} \in \tilde{\mathcal{G}} = \left\{ \tilde{g} \mid \tilde{g} = \sum_{n=0}^{\infty} \partial_y^n \tilde{g}_n(x,y,t),\ \tilde{g}_n \in \mathcal{M}_{2,2} \right\} .$$

In our considerations of the KdV system, we took g in equation (15) to lie in a set that (if we exclude a small, uninteresting subset) has a group structure. Thus there was never a need to consider a system analogous to (32) as well as the system (15), since if g satisfies (15), $\tilde{g} = g^{-1}$ satisfies $\tilde{g}_t = \tilde{g}P$, $\tilde{g}_x = \tilde{g}M$. Here neither $\mathcal{G}$ nor $\tilde{\mathcal{G}}$ have group structure, so these formulations are distinct. Equations (31) and (32) make sense because there is a well-defined natural multiplication of $\mathcal{G}$ (resp.$\tilde{\mathcal{G}}$) on the left (resp.right) by finite order matrix valued linear differential operators in y.

For our purposes it is important to identify the symmetries of the systems (31) and (32). One easily establishes that there is a well-defined natural multiplication of $\mathcal{G}$ (resp. $\tilde{\mathcal{G}}$) on the right (resp.left) by any $g \in \mathcal{G}$ (resp.$\tilde{g} \in \tilde{\mathcal{G}}$) for which g_n (resp.$\tilde{g}_n$) is polynomial in y, for $n = 0, 1, 2, ...$ From this we deduce the following symmetry of (31):

$$g \to gh \tag{33}$$

$$h \in \mathcal{H} = \left\{ h \mid h = \sum_{n=0}^{\infty} h_n(y)\partial_y^n,\ h_n \in \mathcal{M}_{2,2},\ h_n \text{ polynomial in } y \right\}$$

(Further consideration of (32) will be deferred to [9]). The set $\mathcal{H}$ has a natural ring structure; indeed if $\mathcal{M}$ is an arbitrary ring of matrices we can consider the ring of operators

$$\mathcal{H}_{\mathcal{M}} = \left\{ h \mid h = \sum_{n=0}^{\infty} h_n(y)\partial_y^n,\ h_n \in \mathcal{M},\ h_n \text{ polynomial in } y \right\}. \tag{34}$$

For $\mathcal{M} = U(p)$ the Lie algebra obtained by supplying $\mathcal{H}_{\mathcal{M}}$ with the commutator bracket is essentially a classical limit of the W_∞^p algebras studied by Bakas and Kiritsis [10] and Odake and Sano [11]. Note that any subring of $\mathcal{M}$ gives a subring of $\mathcal{H}_{\mathcal{M}}$. Here we focus our attention on the transformation of g_0 under the action (33); we have

$$g_0 \to \sum_{n=0}^{\infty} g_n h_0^{(n)} \tag{35}$$

where $h_0^{(n)}$ denotes the nth y-derivative of h_0; the sum is finite since h_0 is polynomial. We note that an element of $\mathcal{H}$ with any desired h_0 (of degree m) can be constructed by setting

$$\begin{aligned} h &= h_1 h_2 \\ h_1 &= \sum_{n=0}^{m} h_{1n}\partial_y^n \qquad (h_{1n} \text{ independent of } y) \\ h_2 &= y^m \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} ; \end{aligned} \tag{36}$$

it follows that the full symmetry action on g_0 is generated by the transformations

$$\begin{aligned} g_0 &\to g_0 h_{10} \qquad (h_{10} \text{ constant}) \\ g_0 &\to y g_0 + g_1 . \end{aligned} \tag{37}$$

It is maybe unclear whether the action on g_0 is an $\mathcal{H}$ action or an $\mathcal{H}_0$ action, where $\mathcal{H}_0 = \{h \in \mathcal{H} | h_n = 0, n > 0\}$. Because of the appearance of all the g_n in the transformation law (35), there is a full $\mathcal{H}$ action.

We now consider the evolution of g_0 obtained from (31), and the relationship of the entries of g_0 to u. From the second equation of (31) we find we can write

$$g_0 = \begin{pmatrix} \alpha & \beta \\ -\alpha_x & -\beta_x \end{pmatrix} , \tag{38}$$

where α, β are related to u via

$$\begin{aligned} \alpha_{xx} &= \alpha_y + u\alpha \ , \\ \beta_{xx} &= \beta_y + u\beta \ . \end{aligned} \tag{39}$$

From the first equation of (31), we find that α, β evolve via

$$\begin{aligned} \alpha_t &= \tfrac{1}{4}\alpha u_x - \tfrac{1}{2}\alpha_x u - \tfrac{3}{4}\alpha\partial_x^{-1}u_y + \alpha_{xy} \ , \\ \beta_t &= \tfrac{1}{4}\beta u_x - \tfrac{1}{2}\beta_x u - \tfrac{3}{4}\beta\partial_x^{-1}u_y + \beta_{xy} \ . \end{aligned} \tag{40}$$

The evolutions (10) and (40) of course preserve the relations (39). Setting $q = \beta/\alpha$ (assuming α nonzero) we obtain

$$\frac{\alpha_x}{\alpha} = \frac{q_y - q_{xx}}{2q_x} \tag{41}$$

from which

$$u = -\tfrac{1}{2}\left(\frac{q_{xxx}}{q_x} - \frac{3q_{xx}^2}{2q_x^2}\right) + \left(\frac{q_y}{q_x}\right)_x + \tfrac{1}{4}\left(\frac{q_y}{q_x}\right)^2 - \tfrac{1}{2}\partial_x^{-1}\left(\frac{q_y}{q_x}\right)_y \ . \tag{42}$$

q satisfies the evolution equation

$$q_t = \tfrac{1}{4} q_x \left(\left(\frac{q_{xxx}}{q_x} - \frac{3q_{xx}^2}{2q_x^2} \right) + \tfrac{3}{2} \left(\frac{q_y}{q_x} \right)^2 + 3\partial_x^{-1} \left(\frac{q_y}{q_x} \right)_y \right) . \tag{43}$$

This equation has appeared a number of times in the literature (see for example [12] and references therein); the novelty of our approach is that we can write down the infinite dimensional symmetry which leaves the evolution (43) and u as given in (42) invariant. The first of the symmetries in (37) gives the obvious Möbius symmetry

$$q \to \frac{aq+b}{cq+d} \qquad (a, b, c, d \text{ constant}). \tag{44}$$

The second symmetry in (37) becomes

$$q \to \frac{yq+R}{y+S} , \tag{45}$$

where R, S (defined by $\alpha R = (g_1)_{12}$, $\alpha S = (g_1)_{11}$) are functions satisfying

$$\begin{aligned} R_y &= R_{xx} - q - R_x \left(\frac{q_{xx} - q_y}{q_x} \right) \\ S_y &= S_{xx} - 1 - S_x \left(\frac{q_{xx} - q_y}{q_x} \right) \end{aligned} \tag{46}$$

$$\begin{aligned} R_t &= R_{xxx} - \tfrac{3}{2} R_{xx} \left(\frac{q_{xx} - q_y}{q_x} \right) - \tfrac{3}{4} R_x \left(\frac{q_{xxx}}{q_x} - \frac{3q_{xx}^2}{2q_x^2} + \frac{2q_y q_{xx}}{q_x^2} - \tfrac{1}{2} \left(\frac{q_y}{q_x} \right)^2 - \partial_x^{-1} \left(\frac{q_y}{q_x} \right)_y \right) \\ S_t &= S_{xxx} - \tfrac{3}{2} S_{xx} \left(\frac{q_{xx} - q_y}{q_x} \right) - \tfrac{3}{4} S_x \left(\frac{q_{xxx}}{q_x} - \frac{3q_{xx}^2}{2q_x^2} + \frac{2q_y q_{xx}}{q_x^2} - \tfrac{1}{2} \left(\frac{q_y}{q_x} \right)^2 - \partial_x^{-1} \left(\frac{q_y}{q_x} \right)_y \right) . \end{aligned} \tag{47}$$

Formula (42) (or rather the formula for u_x obtained from (42), thereby eliminating the awkward integration symbol on the right hand side) thus gives a remarkable infinite dimensional extension of the standard Schwartzian derivative, with u_x invariant under the symmetries generated by (44) and (45), where R, S satisfy (46). In fact it can be shown that the map (44) is a special case of (45). It would be interesting, but probably rather hard, to prove the uniqueness of this invariant, in the sense of [2]. For the reader concerned by the asymmetry between R and S in (46), we note that (46) and (47) imply identical y and t evolutions for the two functions $R + yq$ and $S + y$, and actually both of these functions satisfy the UrKP equation (43).

Two tasks remain: to understand the place of the MKP equation (9) in our framework, and to obtain Bäcklund transformations. Above, we have defined a projective action of the ring $\mathcal{H}$ (defined in (33)) on q, and we expect the MKP field to be a combination of q and its derivatives, invariant under some subring of $\mathcal{H}$ [3]. Apparently a number of candidate subrings are available, both infinite dimensional (e.g. $\{h|\text{all } h_n \text{ upper triangular}\}$), and finite dimensional (e.g. $\{h|h_n = 0, n > 0, h_0 \text{ constant}\}$), but on reflection the necessary subring must be obtained from $\mathcal{H}$ by placing restrictions only on h_0. At present we do not have a general procedure for constructing an invariant corresponding to any given subring. The one tool we do have for constructing invariants exploits the notion of *gauge symmetry*, which is the invariance of equations (30) and (31) under transformations

$$\begin{aligned} g &\to \bar{g} = sg \\ M &\to \bar{M} = sMs^{-1} - s_x s^{-1} \\ P &\to \bar{P} = sPs^{-1} - s_t s^{-1}. \end{aligned} \tag{48}$$

Here

$$s = \begin{pmatrix} s_1(x,y,t) & 0 \\ s_{\text{op}} & s_2(x,y,t) \end{pmatrix} , \tag{49}$$

where s_{op} is a finite order differential operator in y (with coefficients functions of x, y, t), and s_1, s_2 are functions. This choice of s allows s^{-1} and all the necessary multiplications in (48) to be defined. One can exploit gauge symmetry to bring g to a normal form; but then $\bar{M}, \bar{P}$ will not be invariant under all the symmetries (33), since although M, P are, the gauge transformation s needed to bring g to this normal form is not. However it is reasonable to hope that in fact $\bar{M}, \bar{P}$ will be invariant under some subring of $\mathcal{H}$ (c.f. [7]).

As an example of such a procedure we consider gauge transformations that bring g to a form where $(g_0)_{21} = 0$. Any gauge transformation satisfying

$$s_{\text{op}}|\alpha = s_2 \alpha_x \tag{50}$$

does this, where $s_{\text{op}}|\alpha$ denotes the function obtained by letting s_{op} act on α. One way to satisfy (50) is to take $s_1 = s_2 = 1$, $s_{\text{op}} = \alpha_x/\alpha$. Then writing $j = -2\alpha_x/\alpha = (q_{xx} - q_y)/q_x$ we find

$$\begin{aligned} \bar{M} &= \begin{pmatrix} \frac{1}{2}j & 1 \\ \partial_y + \frac{1}{2}\partial_x^{-1} j_y & -\frac{1}{2}j \end{pmatrix} \\ \bar{P} &= \begin{pmatrix} \frac{1}{2}j\partial_y + \frac{1}{8}f_1 & \partial_y + \frac{1}{4}f_2 \\ \partial_y^2 - \frac{1}{4}f_3\partial_y - \frac{1}{8}f_4 & -\frac{1}{2}j\partial_y - \frac{1}{8}f_5 \end{pmatrix} , \end{aligned} \tag{51}$$

[3] While $\mathcal{H}$ has a ring structure, as do the subsets of $\mathcal{H}$ that will interest us, the additive structure is unimportant for us, so it might be clearer for some to replace the words “ring” and “subring” here and in what follows by the words “monoid” and “submonoid”.

$$
\begin{aligned}
f_1 &= j_{xx} - \tfrac{1}{2}j^3 - j\partial_x^{-1}j_y + 3\partial_x^{-1}(jj_y) + 3\partial_x^{-2}j_{yy} \\
f_2 &= j_x - \tfrac{1}{2}j^2 - \partial_x^{-1}j_y \\
f_3 &= j_x + \tfrac{1}{2}j^2 - \partial_x^{-1}j_y \\
f_4 &= (\partial_x^{-1}j_y)^2 + (j_x + \tfrac{1}{2}j^2)\partial_x^{-1}j_y - 4\partial_x^{-1}j_{yy} \\
f_5 &= j_{xx} - \tfrac{1}{2}j^3 + 2j_y - j\partial_x^{-1}j_y - 3\partial_x^{-1}(jj_y) - 3\partial_x^{-2}j_{yy}.
\end{aligned}
$$

The equation of zero curvature for $\bar{M}, \bar{P}$ gives the MKP equation (9)[4]. The invariance of j is easily found to be the subring of elements of $\mathcal{H}$ with $(h_0)_{11}$ constant and $(h_0)_{21}$ zero. This is a maximal infinite-dimensional subring of $\mathcal{H}$, the action of which on q is generated by the transformations

$$
q \to yq + R \ , \tag{52}
$$

which include Möbius transformations (44) with $c = 0$ as a special case. Transformations (52) are a special case of transformations (45) corresponding to the choice $S = 1 - y$. Under the general transformation (45) one finds

$$
j \to j - \frac{2S_x}{S + y} \ . \tag{53}
$$

We will use this to find a Bäcklund transformation shortly. But first we complete this section on the origin of the MKP equation by noting that the KP-Miura map can be obtained from the definition of j and the first equation of (39), and expresses the fact that any invariant under $\mathcal{H}$ is necessarily an invariant under the subring just given.

In this paper we will only consider the simplest Bäcklund transformations for MKP and KP, deferring a more detailed study, and comparison with the results of [12],[13],[14] for [9]. From above we have a symmetry of MKP given by equation (53), where S satisfies $S_y = S_{xx} - 1 - S_x j$. Similarly, the simpler transformation (44) with $a = d = 0$, $b = c = 1$ gives a symmetry

$$
j \to j - \frac{2q_x}{q} \ . \tag{54}
$$

These symmetries are equivalent, and can be summarized as the Bäcklund transformation

$$
\begin{aligned}
j &\to j - 2q_x/q \\
q_y &= q_{xx} - jq_x \\
q_t &= q_{xxx} - \tfrac{3}{2}jq_{xx} - \tfrac{3}{4}(j_x - \tfrac{1}{2}j^2 + \partial_x^{-1}j_y)q_x \ .
\end{aligned} \tag{55}
$$

---

[4] Note this zero curvature form for MKP is not canonical (in the sense of Drinfeld and Sokolov), unlike its counterpart for MKdV found by replacing ∂_y by λ and setting j_y to zero in $\bar{P}$ and $\bar{M}$.

One can directly check that this is a strong Bäcklund transformation, that the composite of two such transformations is of the same form, and that such transformations leave u invariant, as expected. (55) becomes powerful when used in tandem with the more obvious Bäcklund transformation of MKP, namely $j \to -j$, $y \to -y$. This latter transformation does not leave u invariant, but rather induces the well known linear strong Bäcklund transformation of KP [13]:

$$\begin{aligned} u &\to u - 2(\alpha_x/\alpha)_x \\ \alpha_y &= \alpha_{xx} - u\alpha \\ \alpha_t &= \alpha_{xxx} - \tfrac{3}{2}u\alpha_x - \tfrac{3}{4}(u_x + \partial_x^{-1}u_y)\alpha \end{aligned} \tag{56}$$

(note KP is invariant under $y \to -y$). Now since the square of this "obvious" transformation for MKP is the identity, it is not apparent that one can use it to find more than one solution from a given one; but using the transformation (55) as well allows one (at least in principle) to find chains of solutions of MKP (and thus KP). The combination of applying the obvious transformation and then a transformation of the form (55) is equivalent to one of the two fundamental gauge transformations of [13]; the second arises from consideration of the system (32).

One other Bäcklund transformation has essentially already appeared in this paper, and therefore merits a mention: the transformation (52), with R satisfying the first equations of (46) and (47) is a strong Bäcklund transformation for the UrKP equation (43).

Concluding Remarks

As we have mentioned, we intend in [9] to give a fuller account of the various different notions of "UrKP" that can be found, and of the associated symmetries, modified equations and Bäcklund transformations. Many other issues remain to be addressed, such as setting our results on UrKP in a hamiltonian framework (c.f.[1]) and understanding the relationship of our formulation of MKP with the existing formulations (see [12], [14], [15], [16]). There also remain certain open questions in the UrKdV formalism - it is not yet clear whether Wilson's ideas can be used to understand the existence of all the equations related by Miura maps to KdV (see [5]), and it would also be interesting to see if Wilson's ideas give us insight into the Bäcklund transformations of KdV and MKdV that we have not discussed, namely those with dependence on dimensionful parameters. Possibly the most important direction for further research though involves the *physical application* of Wilson's ideas. The upshot of Wilson's ideas for KdV and their extension for KP is that whenever a modified KdV or KP equation appears in a physical context, then there is a hidden symmetry waiting to be unearthed. This last direction is currently being actively pursued.

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