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Bäcklund Transformations of MKdV and Painlevé Equations

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Abstract

For $N \geq 3$ there are S_N and D_N actions on the space of solutions of the first non-trivial equation in the $SL(N)$ MKdV hierarchy, generalizing the two Z_2 actions on the space of solutions of the standard MKdV equation. These actions survive scaling reduction, and give rise to transformation groups for certain (systems of) ODEs, including the second, fourth and fifth Painlevé equations.

Given a solution j of the MKdV equation

$$j_t = j_{xxx} - \tfrac{3}{2} j^2 j_x \tag{1}$$

we can construct new solutions, $-j$ and $j - \frac{2}{q}$, where q satisfies

$$\begin{aligned} q_x + qj &= 1 \\ q_t + q(j_{xx} - \tfrac{1}{2} j^3) &= (j_x - \tfrac{1}{2} j^2). \end{aligned} \tag{2}$$

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Equations (2) constitute a strong auto-Bäcklund transformation for the MKdV equation, distinct from the usual one given in the literature (see for example [1], Chapter 8, Exercise 2), and discovered, I believe, in the context of Painlevé analysis [2]. If we choose the integration constant arising in the solution of (2) appropriately, the square of this transformation is the identity; but when combined with the $j \to -j$ transformation it can, generically, be used to generate an infinite number of solutions from a particular one (solutions of MKdV that are periodic under the action of the combined symmetry are discussed in [3]).

Unlike the standard auto-Bäcklund transformation for MKdV, the transformations $j \to -j$, $j \to j - \frac{2}{q}$ do not contain dimensionful parameters, and hence survive scaling reduction.[1] Setting $j = -2(3t)^{-\frac{1}{3}} J(w)$ where $w = -x(3t)^{-\frac{1}{3}}$, (1) reduces to the second Painlevé equation (PII)

$$J'' = 2J^3 + Jw + \alpha \tag{3}$$

where a prime denotes differentiation with respect to w and α is an integration constant. Setting $q = (3t)^{\frac{1}{3}} Q(w)$ in (2), we can solve for Q and thus we have two *explicit* transformations for (3),

$$\begin{aligned} &J \to -J, \qquad &&\alpha \to -\alpha \\ &J \to J + \frac{\frac{1}{2} - \alpha}{J' - J^2 - \frac{1}{2}w}, \qquad &&\alpha \to 1 - \alpha. \end{aligned} \tag{4}$$

These transformations, which both square to the identity, generate the well-known transformation group of PII (see [5],[6] and references therein). In solving for Q I have assumed $\alpha \neq \frac{1}{2}$; for $\alpha = \frac{1}{2}$ there is a one parameter family of solutions of PII given by the solutions of $J' = J^2 + \frac{1}{2}w$ (which can be solved in terms of Airy functions).

The purpose of this note is to present generalizations of the above transformations for the SL(N) MKdV equations, $N \geq 3$. The above transformations for the standard MKdV equation actually extend to the MKdV hierarchy, and similarly in the $SL(N)$ case the relevant transformations extend to the hierarchy. But for clarity we will focus just on the lowest nontrivial equation in the hierarchy. On scaling reduction all the transformations become explicit. The (lowest nontrivial) $SL(3)$ and $SL(4)$ MKdV equations reduce, respectively, to the PIV and PV equations (the first of these facts originated, I believe, in [7]; the second is, I believe, new); we recover the transformation groups investigated by Okamoto for PIV [6] (see also [8]) and PV [9].

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[1] When the dimensionful parameter in the standard transformation is set to zero, the transformation becomes trivial. The standard auto-Bäcklund transformation for KdV [1] remains non-trivial even when the dimensionful parameter is set to zero, and this was used in [4], along with the Miura map, to obtain a different derivation of the transformation group of PII from the one we are about to see.

The $SL(N)$ MKdV hierarchy describes evolutions of N fields j_i, $i=1,..,N$, with $\sum_{i=1}^N j_i = 0$. Writing $\Sigma = \frac{1}{N}\sum_{i=1}^N j_i^2$, the lowest nontrivial evolution in the hierarchy is

$$\partial_t j_i = \partial_x \left[\sum_{r=1}^{N-1} \left(1-\tfrac{2r}{N}\right) \partial_x j_{i+r(\mathrm{mod}\ N)} + j_i^2 - \Sigma \right], \qquad i=1,..,N \tag{5}$$

or, equivalently,

$$\partial_t (j_i - j_{i+1}) = \partial_x [\partial_x (j_i + j_{i+1}) + j_i^2 - j_{i+1}^2], \qquad i=1,..,N-1. \tag{5$'$}$$

A simple way to obtain (5) is by reduction of the $SL(N)$ self-dual Yang-Mills equations with an ansatz given in the appendix. (5) has one obvious symmetry group:

Prop.1: D_N Invariance of (5).

Equations (5) are invariant under the D_N action generated by

$$\begin{aligned} A &: j_i \to j_{i+1(\mathrm{mod}\ N)}, x \to x, t \to t \\ B &: j_i \to j_{N+1-i}, x \to -x, t \to -t \end{aligned} \tag{6}$$

which satisfy $A^N = B^2 = I$, $ABAB = I$.

The other symmetry group is less obvious:

Prop.2: S_N Invariance of (5).

There is an S_N action on solutions of (5); the action of the fundamental transpositions $T_i = (i\ i+1)$, $i=1,..,N-1$, is given by

$$\begin{aligned} j_i &\to j_i + q_i^{-1} \\ j_{i+1} &\to j_{i+1} - q_i^{-1} \\ j_r &\to j_r, \quad r \neq i, i+1 \end{aligned} \tag{7}$$

where q_i satisfies[2]

$$\begin{aligned} & q_{ix} + (j_i - j_{i+1}) q_i + 1 = 0 \\ & q_{it} + [(j_i + j_{i+1})_x + j_i^2 - j_{i+1}^2] q_i + (j_i + j_{i+1}) = 0. \end{aligned} \tag{8}$$

For a complete understanding of the origin of these transformations, and why the group generated by the transformations T_i is S_N, the reader is referred to [10]. The basic

[2] Equations (8) determine q_i up to a parameter. The precise action of T_i is determined by picking a suitable boundary condition satisfied by the solution $j_1, ..., j_N$ on which we wish to act, and requiring the T_i to preserve this condition. For example, for the scaling reduction of (5) we will shortly consider, we fix this parameter by requiring that q_i should also have a well-defined scaling behavior.

argument however is quite simple: consider the Nth order differential operator $L = (\partial + j_N)..(\partial + j_2)(\partial + j_1)$, and choose a basis $\{\psi_1, .., \psi_N\}$ for the kernel of L such that $\{\psi_1, .., \psi_i\}$ is a basis for the kernel of $(\partial + j_i)..(\partial + j_2)(\partial + j_1)$ for each i, $i = 1, .., N$. It is easy to check that switching ψ_i and ψ_{i+1} induces the change in the j_r given by (7), with q_i satisfying the first equation in (8); the second equation in (8) is then deduced directly from (5).

There are relations between the D_N and S_N generators; two obvious ones are

$$\begin{aligned} AT_i &= T_{i-1}A, \qquad i = 2, 3, .., N-1 \\ BT_i &= T_{N-i}B, \qquad i = 1, 2, .., N-1. \end{aligned} \tag{9}$$

It is natural to define a transformation T_N by $T_N \equiv A^{-1}T_{N-1}A$; this satisfies $AT_1 = T_N A$ and $BT_N = T_N B$. The explicit action of T_N is

$$\begin{aligned} j_N &\to j_N + q_N^{-1} \\ j_1 &\to j_1 - q_N^{-1} \\ j_r &\to j_r, \quad r \neq N, 1 \end{aligned} \tag{10}$$

where q_N satisfies

$$\begin{aligned} &q_{Nx} + (j_N - j_1)q_N + 1 = 0 \\ &q_{Nt} + [(j_N + j_1)_x + j_N^2 - j_1^2]q_N + (j_N + j_1) = 0. \end{aligned} \tag{11}$$

It must be emphasized that T_N is *not* a pure S_N transformation, and should not be confused with the fundamental transposition $(1\ N)$ in S_N, which generically changes all the j_i. Having introduced T_N, it is clear that the transformation group for (5) is a semi-direct product of the group generated by $T_1, .., T_N$ with the group D_N.

We now consider the scaling reduction of (5). Writing $j_i = t^{-\frac{1}{2}}J_i(w)$ where $w = t^{-\frac{1}{2}}x$, we find that we can at once integrate each equation of (5) to obtain the reduced system

$$-\tfrac{1}{2}wJ_i + \alpha_i = \sum_{r=1}^{N-1} \left(1 - \tfrac{2r}{N}\right) J'_{i+r(\mathrm{mod}\ N)} + J_i^2 - S, \qquad i = 1, .., N. \tag{12}$$

Here $S = \frac{1}{N}\sum_{r=1}^N J_r^2$, a prime denotes differentiation with respect to w,the α_i, $i = 1, .., N$, are constants satisfying $\sum_{r=1}^N \alpha_i = 0$, and $\sum_{r=1}^N J_i = 0$. Because of the square roots in the reduction formulae, (12) displays a residual scale invariance $w \to -w$, $J_i \to -J_i$.[3] This can be eliminated by setting $J_i(w) = w^{-1}K_i(z)$, where $z = w^2$, to obtain the system

$$(\alpha_i - \tfrac{1}{2}K_i)z = \sum_{r=1}^{N-1} \left(1 - \tfrac{2r}{N}\right) \left(2z\dot{K}_{i+r(\mathrm{mod}\ N)} - K_{i+r(\mathrm{mod}\ N)}\right) + K_i^2 - T, \qquad i = 1, .., N. \tag{13}$$

[3] Similar considerations give rise to the extra invariances of equation (3) under $J \to \lambda J$, $w \to \lambda^2 w$ with $\lambda^3 = 1$.

Here a dot denotes differentiation with respect to z, $T = \frac{1}{N}\sum_{r=1}^{N} K_i^2$, and $\sum_{r=1}^{N} K_i = 0$. We could, of course, have obtained (13) directly from (5) by substituting $j_i = x^{-1}K_i(z)$ where $z = t^{-1}x^2$, but if we do this it is somewhat harder to see the integrations that can be done.

Under scaling reduction (i.e. setting $q_i = xQ_i(z)$) we find we can solve (8) for q_i; we can thus write down both the D_N and S_N actions explicitly:

Prop.1′: D_N Invariance of (13).

Equations (13) are invariant under the D_N action generated by

$$\begin{aligned} &A : K_i \to K_{i+1(\mathrm{mod}\ N)}, z \to z, \alpha_i \to \alpha_{i+1(\mathrm{mod}\ N)} \\ &B : K_i \to -K_{N+1-i}, z \to -z, \alpha_i \to -\alpha_{N+1-i} \end{aligned} \tag{14}$$

which satisfy $A^N = B^2 = I$, $ABAB = I$.

Prop.2′: S_N Invariance of (13).

There is an S_N action on solutions of (13); the action of the fundamental transpositions $T_i = (i\ i+1)$, $i = 1,..,N-1$, is given by

$$\begin{aligned} K_i &\to K_i + \frac{z(\alpha_{i+1} - \alpha_i - \frac{1}{2})}{K_i + K_{i+1} + \frac{1}{2}z} \\ K_{i+1} &\to K_{i+1} - \frac{z(\alpha_{i+1} - \alpha_i - \frac{1}{2})}{K_i + K_{i+1} + \frac{1}{2}z} \\ K_r &\to K_r, \quad r \neq i, i+1 \\ \alpha_i &\to \alpha_{i+1} - \tfrac{1}{2} \\ \alpha_{i+1} &\to \alpha_i + \tfrac{1}{2} \\ \alpha_r &\to \alpha_r, \quad r \neq i, i+1. \end{aligned} \tag{15}$$

The transformations T_i can easily be checked using the reduced form of $(5)'$ (equivalent to (13)):

$$2z(\dot{K}_i+\dot{K}_{i+1}) = (K_{i+1}+K_i)(K_{i+1}-K_i+1)+\tfrac{1}{2}z(K_{i+1}-K_i)+z(\alpha_i-\alpha_{i+1}), \qquad i = 1,..,N-1. \tag{13$'$}$$

For comparison with [6],[9] it is useful to define $\beta_i \equiv N^{-1}(2\alpha_i - i + \frac{1}{2}(N+1))$; the action of T_i, $i = 1,..,N-1$, on the β_r is $\beta_i \to \beta_{i+1}$, $\beta_{i+1} \to \beta_i$, and $\beta_r \to \beta_r$ for $r \neq i, i+1$. The action of T_N is $\beta_1 \to \beta_N + 1$, $\beta_N \to \beta_1 - 1$, and $\beta_r \to \beta_r$ for $r \neq 1, N$. The action of A is $\beta_r \to \beta_{r+1} + \frac{1}{N}$ for $r \neq N$, and $\beta_N \to \beta_1 - \frac{N-1}{N}$, and the action of B is $\beta_r \to -\beta_{N+1-r}$. The transformation $AT_1T_2...T_{N-1}$ acts as a "parallel transformation", mapping $\vec{\beta}$ to $\vec{\beta} + \frac{1}{N}(1,1,..,1,1-N)$.

We now relate the above systems for $N = 3, 4$ to PIV and PV, and discuss the relevant transformation groups. The following results are elementary to establish.

Prop.3
The general solution of (13) for $N=3$ is

$$\begin{aligned} K_3 &= \frac{M+z}{2} \\ K_2 - K_1 &= \frac{2z\dot{M}}{M} - 1 + \frac{z(1+2\alpha_1-2\alpha_2)}{M} \end{aligned} \tag{16}$$

where $M(z)$ solves the equation

$$\ddot{M} = \frac{\dot{M}^2}{2M} + \frac{3M^3}{32z^2} + \frac{3M^2}{8z} + \left(\frac{3}{4} - \frac{2\alpha_3}{z} - \frac{1}{z^2}\right)\frac{3M}{8} - \frac{(\frac{1}{2}+\alpha_1-\alpha_2)^2}{2M}. \tag{17}$$

$M(z)$ solves (17) if and only if $J(p)$ defined by $M(z)=2pJ(p)$, where $p=(3z/4)^{\frac{1}{2}}$, satisfies PIV:

$$\frac{d^2J}{dp^2} = \frac{1}{2J}\left(\frac{dJ}{dp}\right)^2 + \tfrac{3}{2}J^3 + 4pJ^2 + 2(p^2-2\alpha_3)J - \frac{2}{J}\left(\frac{1+2\alpha_1-2\alpha_2}{3}\right)^2. \tag{18}$$

The transformation group for (17) is a semi-direct product of the group generated by T_1, T_2, T_3, which Okamoto [6] calls $s_1, s_2, \tilde{s}$, with the D_N group generated by A, B. Okamoto writes $\tilde{l}$ for A^{-1} (and l for $T_2T_1A^{-1}$), and instead of B uses $x = AB$ (all this can easily be checked; Okamoto's coefficients v_1, v_2, v_3 are the coefficients $\beta_1, \beta_2, \beta_3$ introduced above). The transformation group for (18) is just that for (17) supplemented with the extra symmetry $J \to -J$, $p \to -p$, which Okamoto denotes ψ. Explicit formulae for all the transformations can easily be written.

Prop.4
The general solution of (13) for $N=4$ is

$$\begin{aligned} K_1 &= \frac{z\dot{V}}{V(V-1)} + \frac{z}{4}\frac{V+1}{V-1} - \frac{V+1+2(V-1)(\alpha_2-\alpha_1)}{4V} \\ K_2 &= -\frac{z\dot{V}}{V(V-1)} + \frac{z}{4}\frac{V+1}{V-1} + \frac{V+1+2(V-1)(\alpha_2-\alpha_1)}{4V} \\ K_3 &= \frac{z\dot{V}}{(V-1)} - \frac{z}{4}\frac{V+1}{V-1} - \frac{V+1+2(V-1)(\alpha_3-\alpha_4)}{4} \end{aligned} \tag{19}$$

where $V(z)$ solves PV:

$$\begin{aligned} \ddot{V} = \left(\frac{1}{2V} + \frac{1}{V-1}\right)\dot{V}^2 - \frac{\dot{V}}{z} - \frac{V(V+1)}{8(V-1)} + \frac{(\alpha_1+\alpha_2)V}{2z} \\ + \frac{(V-1)^2}{32z^2}\left((1+2\alpha_3-2\alpha_4)^2V - \frac{(1+2\alpha_1-2\alpha_2)^2}{V}\right). \end{aligned} \tag{20}$$

Note that for $N \neq 4$ the order of the system (13) is $N-1$, but for $N = 4$ it is 2. The form of PV in (20) is brought to the standard form of Okamoto [9] by rescaling z. Having done this, it is straightforward to check that the coefficients $\beta_1, .., \beta_4$ are exactly the coefficients $v_1, .., v_4$ of Okamoto. The relationship between the transformations we have introduced and those of [9] is as follows. First we note that since V is determined by $K_1 + K_2$, the transformations T_1, T_3 leave V unchanged. These are π_1' and π_1 in [9], respectively. s_1, s_2, s_3, s_0 in [9] are T_1, T_2, T_3, T_4 respectively, and l is $T_3T_2T_1A^{-1}$. Finally x (or π_2) of [9] is BA^2, and w' of [9] is T_1T_3B.

Discussion

It is pleasing that we have obtained the results of [6] and [9] in a unified and extended framework; we see the rather complicated transformation groups for PIV and PV have a fairly simple origin in the symmetries of (5). It is to be expected that useful applications will be found for the system (13) for $N \geq 5$, and for the systems obtained by scaling reduction of higher equations in the MKdV hierarchies (all of these systems possess the Painlevé property). For example, in generalization of the results of [11], we should expect the scaling reductions of equations in the $SL(N)$ MKdV hierarchy to arise as the "string equations" for certain matrix models.

One thing missing from this paper is an explanation of the origin of the transformation groups for PIII [12] and PVI [13]. From Okamoto's work on these systems one might guess that they arise as scaling reductions of an MKdV equation associated with the Lie algebras B_2 and D_4 respectively [14]. The lowest nontrivial flow in the B_2 MKdV hierarchy (which describes the evolution of two fields j_1, j_2) can be computed, and has a consistent scaling reduction; each of the resulting pair of equations can be integrated once, as above, but the remaining system is a *fourth* order system with two arbitrary constants. Remarkably, this system can actually be written as a single fourth order ODE. But so far I have been unable to establish any connection between this equation and PIII. From [5] it is clear that PIII and PVI have to be discussed in tandem with other systems, so it might not be surprising if they arose naturally embedded in some larger system; but currently the existence of a relation between the B_2 MKdV and PIII remains conjecture.

Another possibility for the origin of the transformation groups of PIII and PVI is that these equations might arise as scaling reductions of some other bihamiltonian integrable system in $1+1$ dimensions (it is known that PIII and PVI arise as reductions of certain $1+1$ dimensional systems - see [15], p.343 for references - but not as scaling reductions). It can be shown that group actions which survive scaling reduction exist on the spaces of solutions of other bihamiltonian systems. Indeed the reader can check that PIV arises as a scaling reduction of the system

$$
\begin{aligned}
j_t &= (j_x + j^2 - 2j\bar{j})_x \\
\bar{j}_t &= (-\bar{j}_x - \bar{j}^2 + 2j\bar{j})_x
\end{aligned}
\tag{21}
$$

which is intimately related to the nonlinear Schrödinger equation (see [16]). Simple Bäcklund transformations of (21), such as $j \to \bar{j}, \bar{j} \to j, x \to x, t \to -t$ generate at least some of the transformations for PIV (compare [17] where some of the transformations for PIV were obtained directly from Bäcklund transformations of NLS).

As a final comment, I note that a new derivation of all the Painlevé equations has recently been given by Mason and Woodhouse, who examined certain symmetry reductions of the $SL(2)$ self-dual Yang-Mills equations [18]. It would be very interesting to see if the transformation groups had an explanation from this viewpoint as well.

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Appendix: Derivation of (5)

The self-dual Yang-Mills equations can be written $F_{\bar{x}\bar{t}} = 0$, $F_{x\bar{t}} = F_{t\bar{x}}$, $F_{xt} = 0$, where $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + [A_\mu, A_\nu]$ $(\mu, \nu \in \{x, t, \bar{x}, \bar{t}\})$ and the A_μ are the "potentials" i.e. Lie-algebra valued functions of $x, t, \bar{x}, \bar{t}$. Equations (5), up to a rescaling of t, are obtained from the $SL(N)$ self-dual Yang-Mills equations with an ansatz:

$$
\begin{aligned}
(A_{\bar{x}})_{ij} &= \delta_{i,j+(N-1)} \\
(A_{\bar{t}})_{ij} &= f\delta_{i,j+(N-1)} - \delta_{i,j+(N-2)} \\
(A_x)_{ij} &= j_i\delta_{i,j} + \delta_{i,j-1} \\
(A_t)_{ij} &= A_i\delta_{i,j} + B_i\delta_{i,j-1} - \delta_{i,j-2}
\end{aligned}
$$

Here the A_i $(i = 1, .., N)$, B_i $(i = 1, .., N-1)$, j_i $(i = 1, .., N)$ and f are functions of x, t alone, with $\sum_{i=1}^N j_i = 0$ and $\sum_{i=1}^N A_i = 0$.

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